4. Symmetrical and Unsymmetrical Bending

- 4.1 Internal Forces in Beams
- 4.2 Simple Formula for Normal Stresses Due to Pure (or Transverse) Bending
- 4.3 Unsymmetric Bending of Straight Beams
- 4.4 Case of Combined Normal Force and Bending Moments
  - Assumption of Plane Cross Section Remaining Plane
- 4.5 Calculation of Displacements
  - Case of Transverse Bending

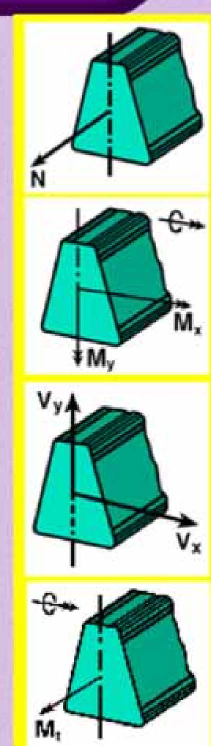
Internal Forces in Beams

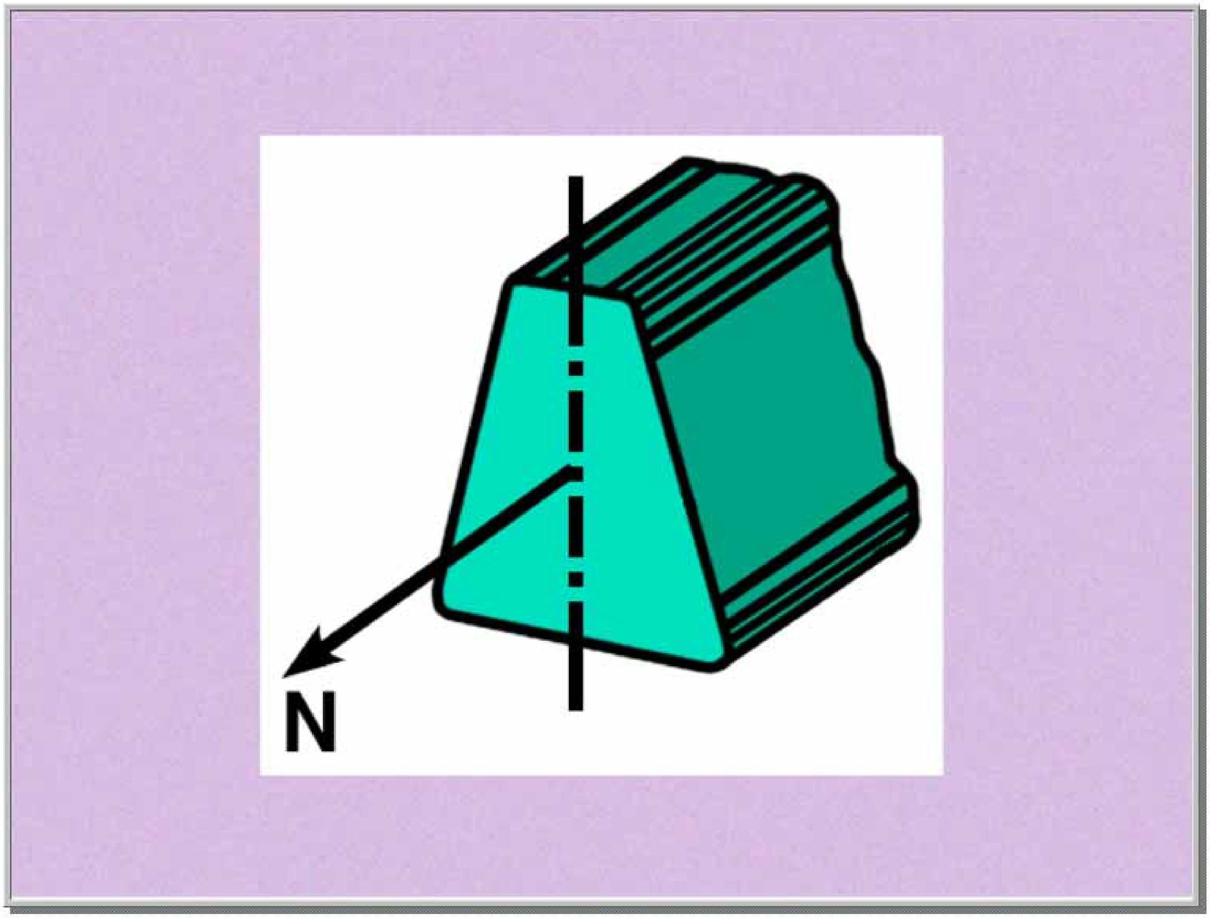
Forces Associated with Normal Stresses

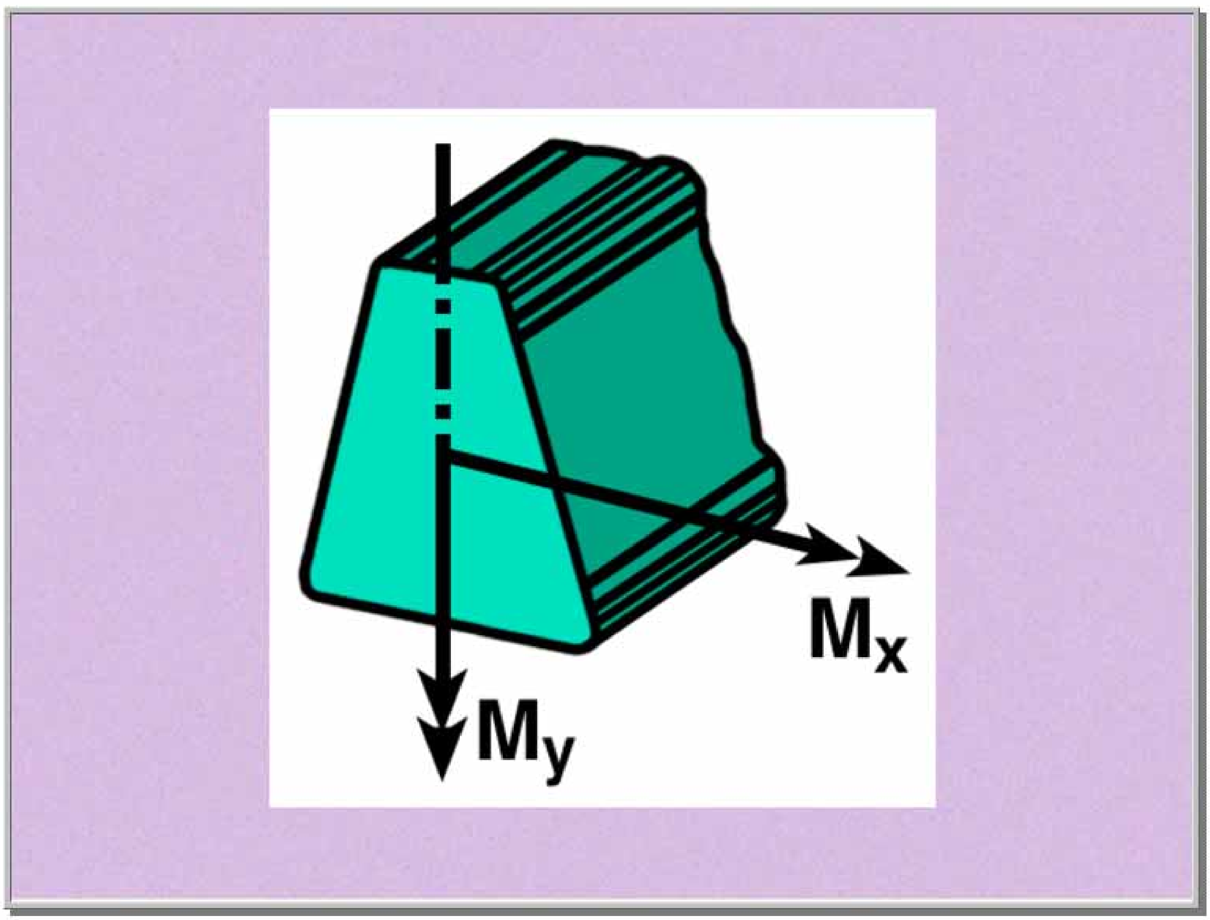
Normal (or axial) force	N
Bending moment (plane yz)	M_x
Bending moment (plane xz)	M_y

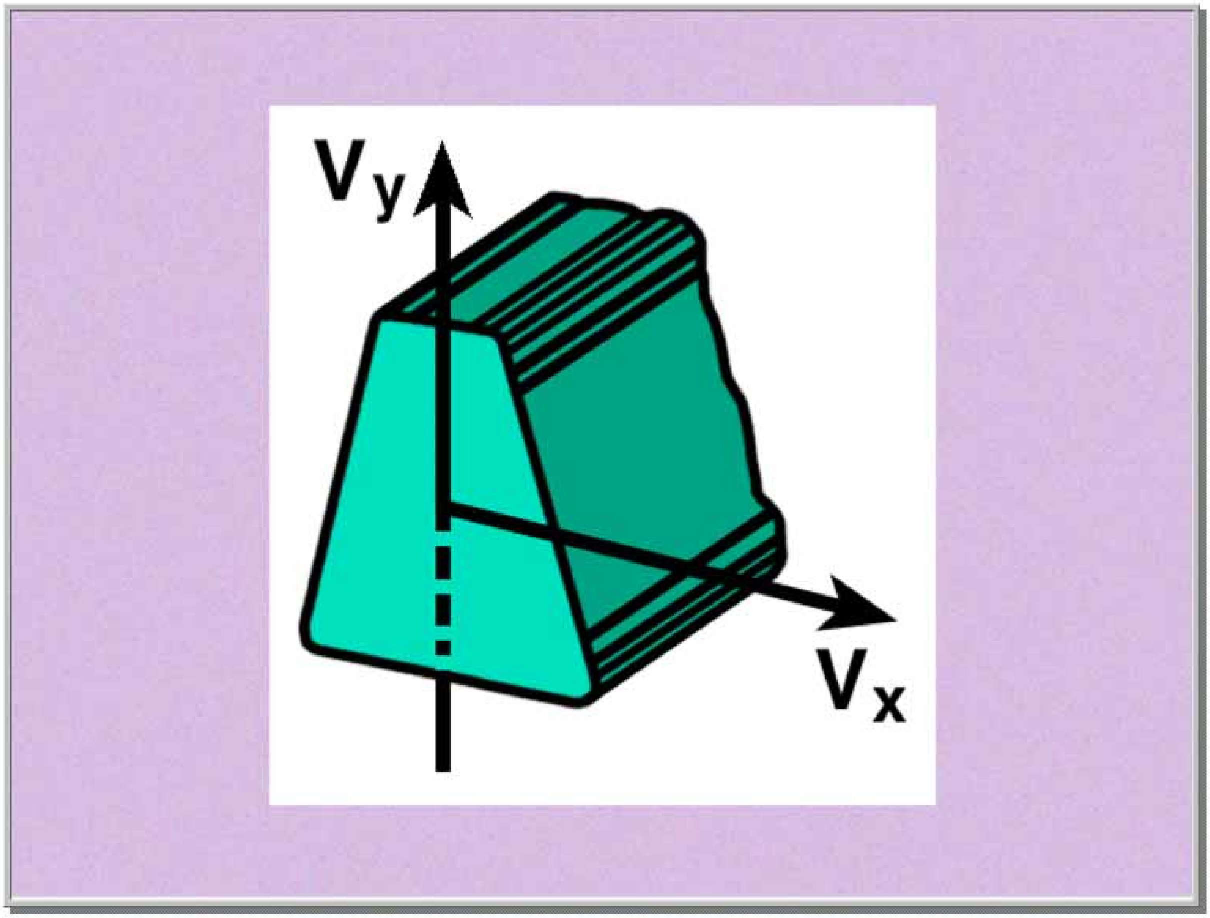
Forces Associated with Shear Stresses

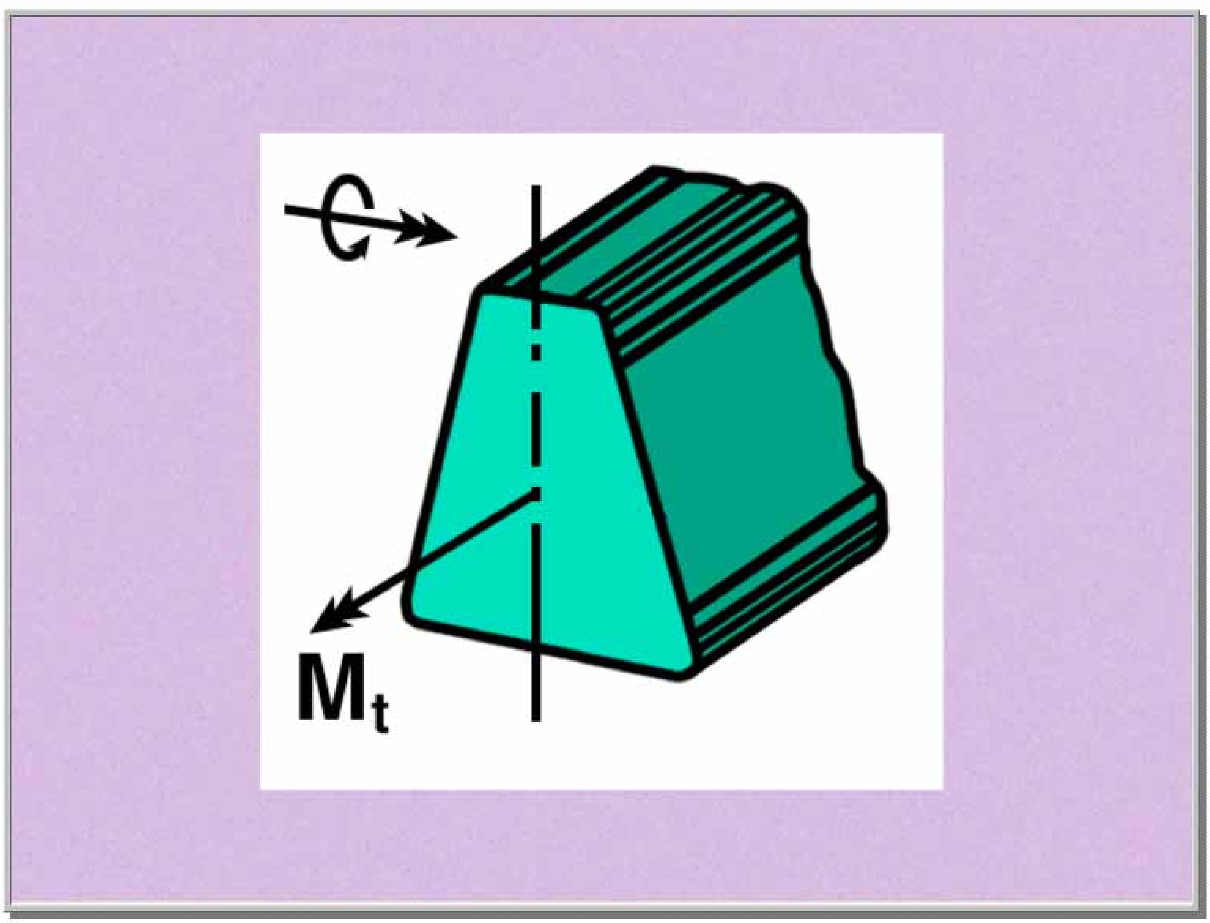
Shearing force	V_x
Shearing force	V_y
Twisting moment (plane xy)	M_t




N

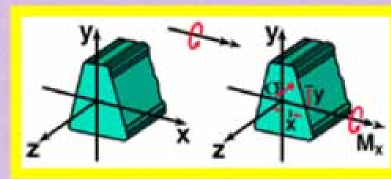

Mx
My


V
y
V
x


M
t

Simple Formula for Normal Stresses Due to Pure (or Transverse) Bending

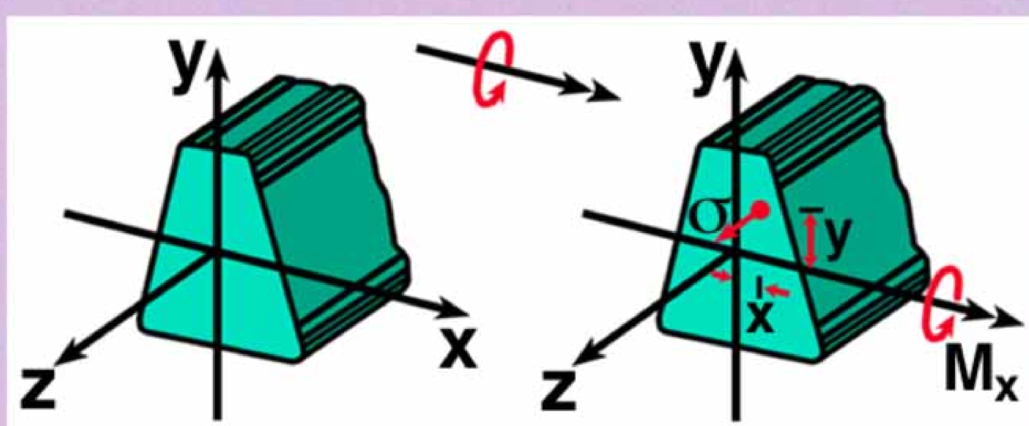
$$\sigma = \frac{M_x\, y}{I_x}$$



Assumptions

- Plane of loads (and of bending) is perpendicular to neutral axis
- x axis is the neutral axis and is a principal centroidal axis

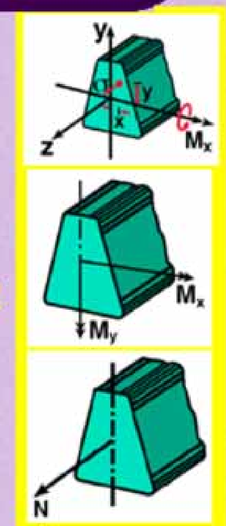
$$S_x = \int_A y\, dA = 0 \;,\; I_{xy} = \int_A xy\, dA = 0$$



Simple Formula for Normal Stresses Due to Pure (or Transverse) Bending

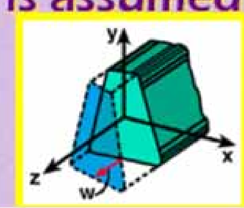
Internal Forces at the Cross Section

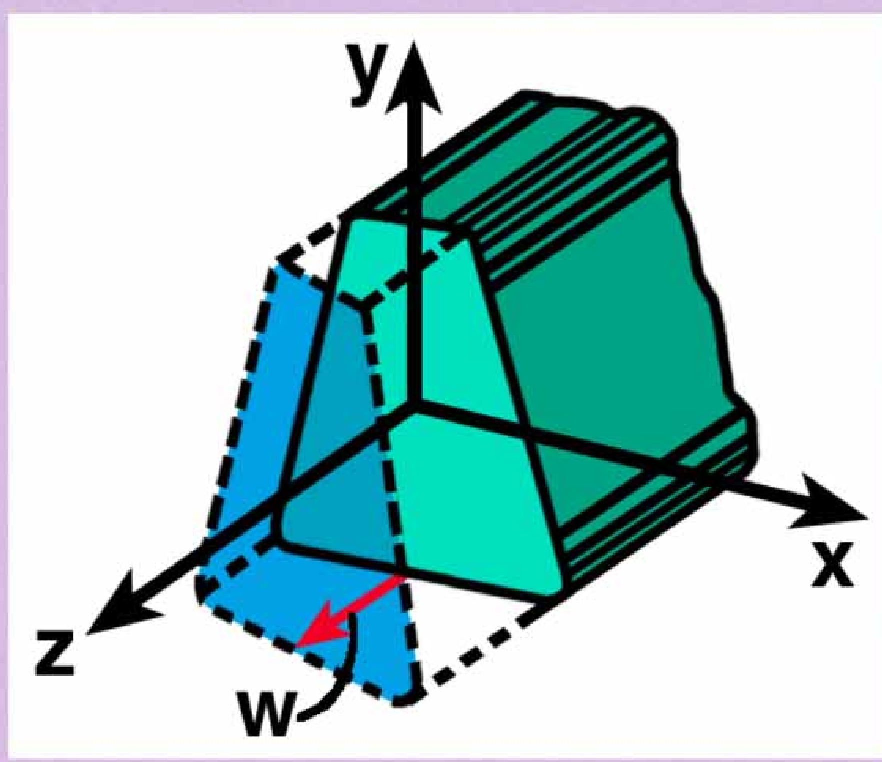
- Normal force N
- Bending moments M_x and M_y about the x and y axes



Kinematic Relations

- Plane cross section before deformation is assumed to remain plane after deformation



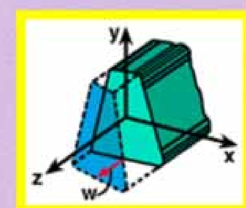


Simple Formula for Normal Stresses Due to Pure (or Transverse) Bending

Kinematic Relations

- Plane cross section before deformation is assumed to remain plane after deformation
- Both the displacement w in the axial direction and the strain ε are linear functions of x and y

$$\varepsilon = \frac{\partial w}{\partial z} = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$$



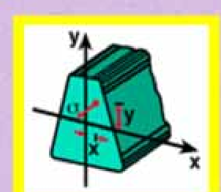
a, b, c are independent of x and y

Simple Formula for Normal Stresses Due to Pure (or Transverse) Bending

Static Relations

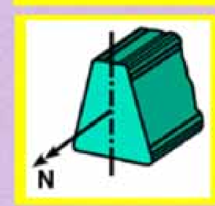
- Sum of internal stresses in the axial direction

$$\int_A \sigma \, dA = N$$



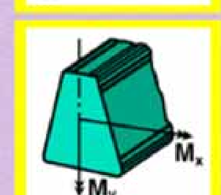
- Sum of moments of internal stresses about y axis

$$\int_A \sigma \, x \, dA = M_y$$



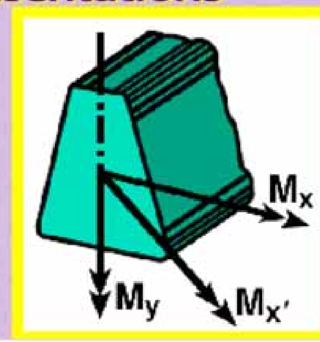
- Sum of moments of internal stresses about x axis

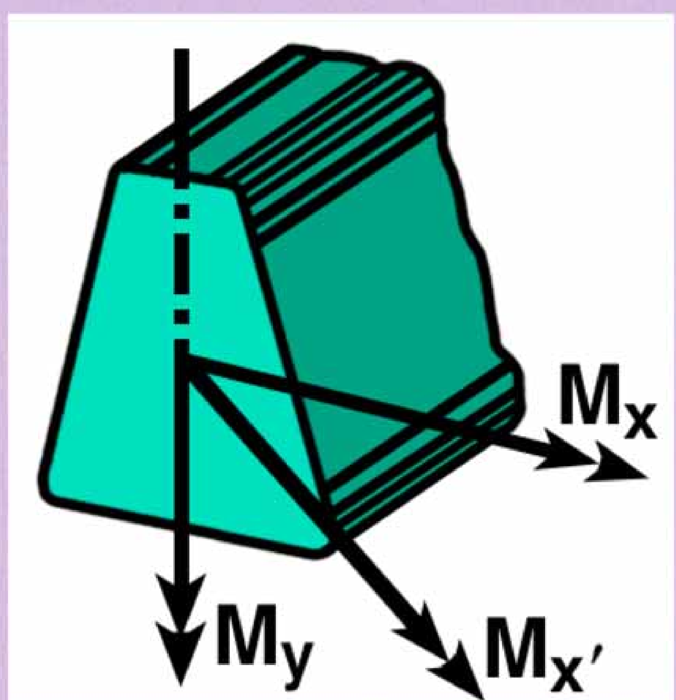
$$\int_A \sigma \, y \, dA = M_x$$



Unsymmetric Bending of Straight Beams

- When the plane of bending is not a principal plane, then
  - Either use a more general formula for the normal stresses (resulting from a bending moment $M_{x'}$), or
  - Decompose the bending moment into components whose vector representations are along principal centroidal axes



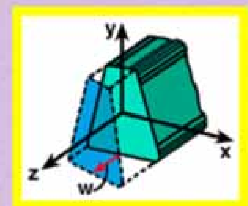


Case of Combined Normal Force and Bending Moments

Constitutive Relations

- For linearly elastic material - uniaxial stress state

$$\sigma = E\,\varepsilon = E\begin{bmatrix}1 & x & y\end{bmatrix}\begin{Bmatrix}a\\ b\\ c\end{Bmatrix}$$



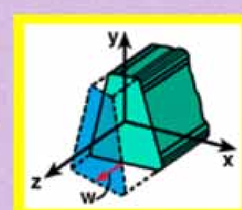
- From the static relations

$$E\begin{bmatrix}A & S_y & S_x\\ S_y & I_y & I_{xy}\\ S_x & I_{xy} & I_x\end{bmatrix}\begin{Bmatrix}a\\ b\\ c\end{Bmatrix} = \begin{Bmatrix}N\\ M_y\\ M_x\end{Bmatrix}$$

Case of Combined Normal Force and Bending Moments

- From the static relations

$$E\begin{bmatrix}A & S_y & S_x\\ S_y & I_y & I_{xy}\\ S_x & I_{xy} & I_x\end{bmatrix}\begin{Bmatrix}a\\ b\\ c\end{Bmatrix} = \begin{Bmatrix}N\\ M_y\\ M_x\end{Bmatrix}$$



where

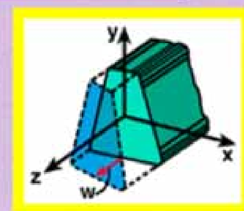
$$\begin{Bmatrix}A\\ S_x\\ S_y\end{Bmatrix} = \int_A \begin{Bmatrix}1\\ y\\ x\end{Bmatrix} dA\ ; \quad \begin{Bmatrix}I_x\\ I_y\\ I_{xy}\end{Bmatrix} = \int_A \begin{Bmatrix}y^2\\ x^2\\ xy\end{Bmatrix} dA$$

Case of Combined Normal Force and Bending Moments

where

$$\begin{Bmatrix} A \\ S_x \\ S_y \end{Bmatrix} = \int_A \begin{Bmatrix} 1 \\ y \\ x \end{Bmatrix} dA \; ; \quad \begin{Bmatrix} I_x \\ I_y \\ I_{xy} \end{Bmatrix} = \int_A \begin{Bmatrix} y^2 \\ x^2 \\ xy \end{Bmatrix} dA$$

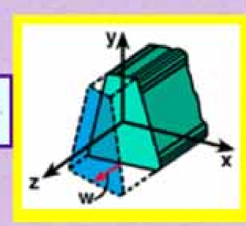
or



$$\begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} A & S_y & S_x \\ S_y & I_y & I_{xy} \\ S_x & I_{xy} & I_x \end{bmatrix}^{-1} \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$

Case of Combined Normal Force and Bending Moments

$$\begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} A & S_y & S_x \\ S_y & I_y & I_{xy} \\ S_x & I_{xy} & I_x \end{bmatrix}^{-1} \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$



- The general formula for normal stresses in terms of the normal force and bending moments:

$$\sigma = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} A & S_y & S_x \\ S_y & I_y & I_{xy} \\ S_x & I_{xy} & I_x \end{bmatrix}^{-1} \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$

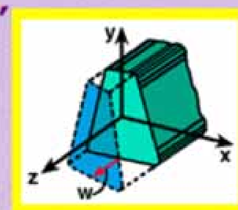
Case of Combined Normal Force and Bending Moments

Simplifications

- If x and y are centroidal axes, then
  $S_x = S_y = 0$

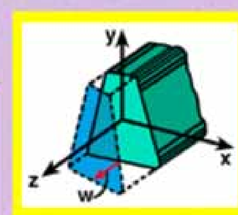
$$\sigma = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} A & 0 & 0 \\ 0 & I_y & I_{xy} \\ 0 & I_{xy} & I_x \end{bmatrix}^{-1} \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$

- If x and y are centroidal principal axes, then $S_x = S_y = 0$, $I_{xy} = 0$



Case of Combined Normal Force and Bending Moments

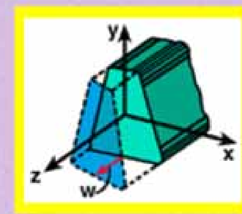
- If x and y are centroidal principal axes, then $S_x = S_y = 0$, $I_{xy} = 0$



$$\sigma = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} \frac{1}{A} & \bullet & \bullet \\ \bullet & \frac{1}{I_y} & \bullet \\ \bullet & \bullet & \frac{1}{I_x} \end{bmatrix} \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$

$$\sigma = \frac{N}{A} + \frac{M_y}{I_y} x + \frac{M_x}{I_x} y$$

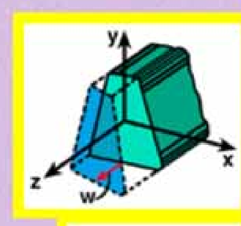
Case of Combined Normal Force and Bending Moments



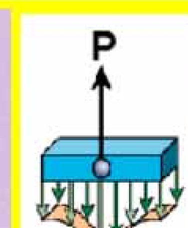
$$\sigma = \frac{N}{A} + \frac{M_y}{I_y}x + \frac{M_x}{I_x}y$$

Neutral Axis
Is the axis at which the normal stress $\sigma = 0$

Case of Combined Normal Force and Bending Moments



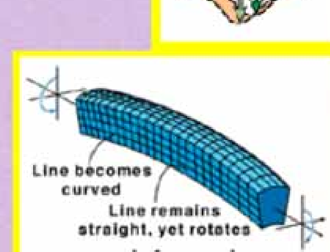
- Assumption of plane cross section before deformation remains plane
  - Is accurate for:
    - axial load (away from point of application of load)
    - pure bending
  - Is approximate for transverse bending

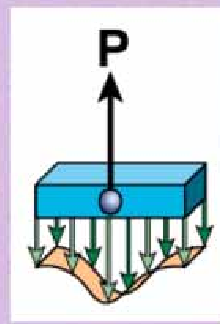


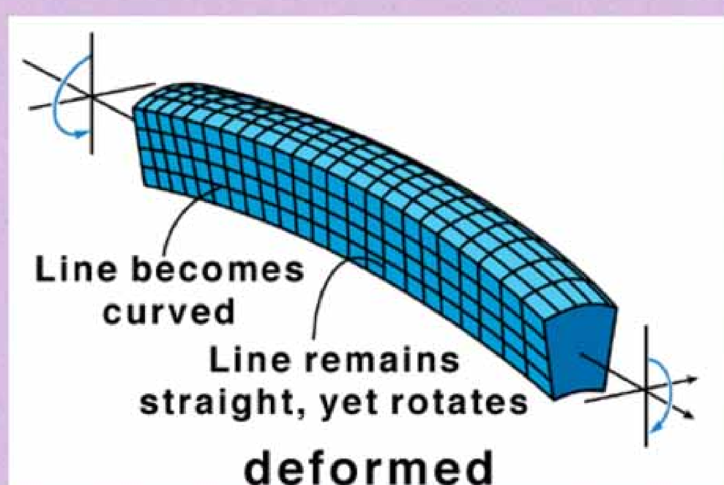
- Assumption of undeformable cross sections

$$\varepsilon_x = \varepsilon_y = \gamma_{xy} = 0$$

- Only approximate for bending



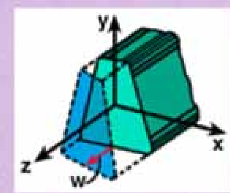

P


Line becomes
curved
Line remains
straight, yet rotates
deformed

Case of Combined Normal Force and Bending Moments

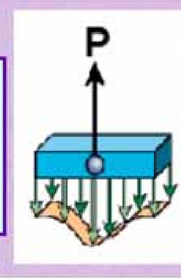
- Assumption of undeformable cross sections

$$\varepsilon_x = \varepsilon_y = \gamma_{xy} = 0$$



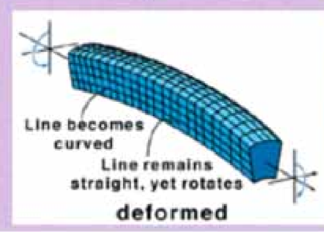
- Only approximate for bending

$$\left(\varepsilon_x\ ,\ \varepsilon_y\ ,\ \gamma_{xy}\right) << \varepsilon_z$$



- More approximate than for beams under axial loading

$$\varepsilon_x = \varepsilon_y = -\upsilon\varepsilon_z$$



Calculations of Displacements

Rotation, Curvature and Axial Strain

- For the case of a single bending moment Mx

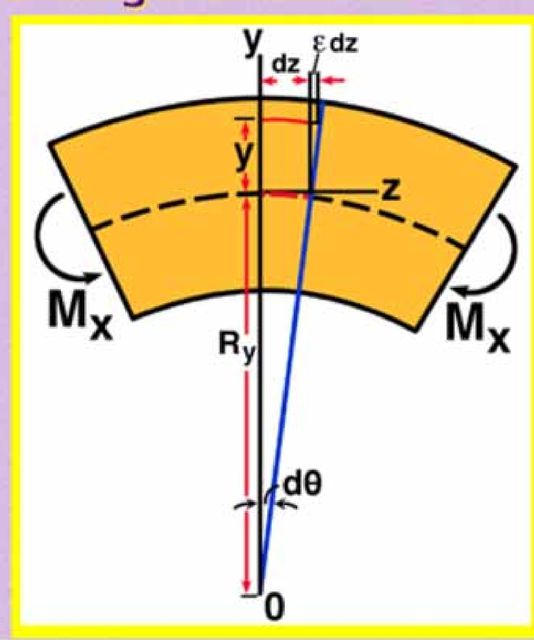
$$d\theta = \frac{dz}{R_y} = \frac{\varepsilon\, dz}{y}$$

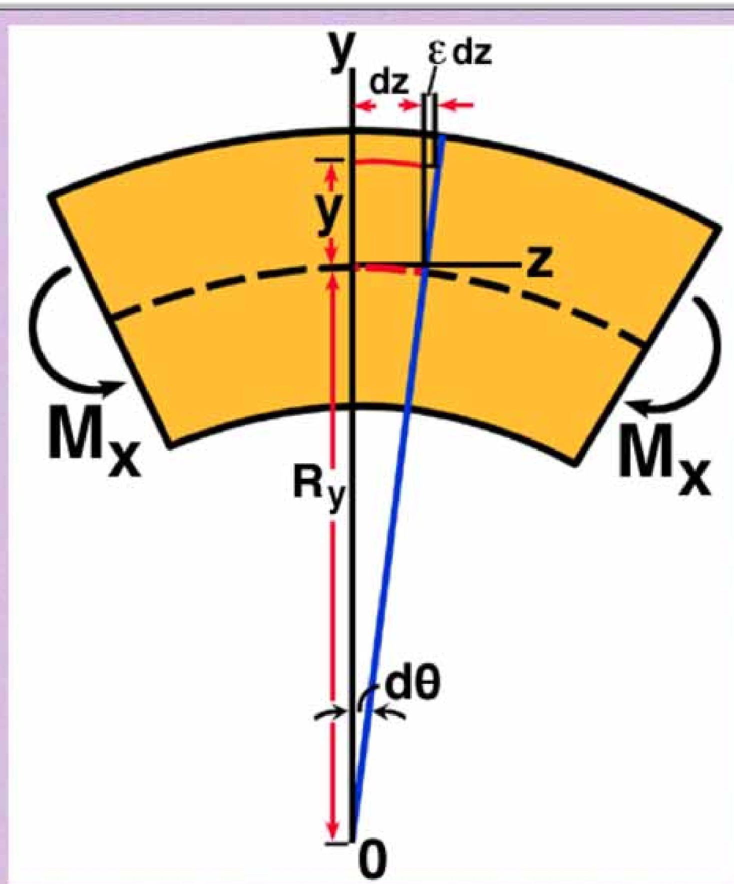
$$\frac{1}{R_y} = \frac{d\theta}{dz} = \frac{\varepsilon}{y}$$

$$\simeq -\frac{d^2v}{dz^2}$$

or

$$\varepsilon \simeq -y\,\frac{d^2v}{dz^2}$$





Calculations of Displacements

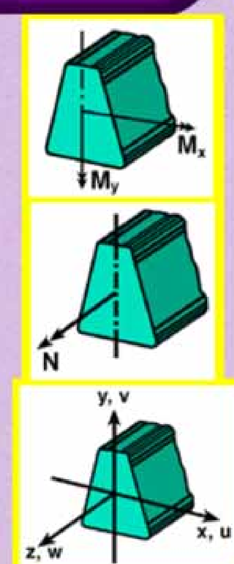
- *Analogously, for a bending moment M_y*

$$\boxed{\varepsilon \simeq -x\,\frac{d^2u}{dz^2}}$$

- *And, for an axial force N* $\boxed{\varepsilon = \varepsilon_o}$

- *For the case of combined axial force, bending moment M_x and bending moment M_y*

$$\varepsilon = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} \varepsilon_o \\ -\dfrac{d^2u}{dz^2} \\ -\dfrac{d^2v}{dz^2} \end{Bmatrix}$$

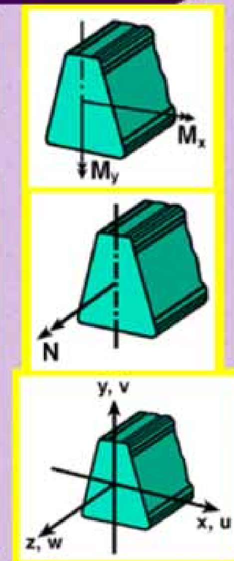


Calculations of Displacements

Displacement equations $\varepsilon = \frac{dw}{dz} = \frac{\sigma}{E}$

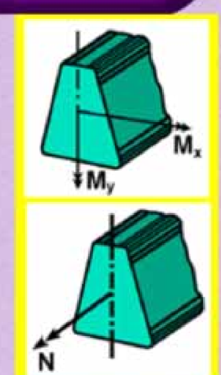
from which

$$\begin{Bmatrix} \varepsilon_o \\ -\frac{d^2u}{dz^2} \\ -\frac{d^2v}{dz^2} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} A & S_y & S_x \\ S_y & I_y & I_{xy} \\ S_x & I_{xy} & I_x \end{bmatrix}^{-1} \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$



Calculations of Displacements

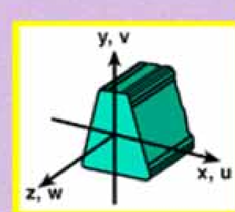
$$\begin{Bmatrix} \varepsilon_o \\ -\frac{d^2u}{dz^2} \\ -\frac{d^2v}{dz^2} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} A & S_y & S_x \\ S_y & I_y & I_{xy} \\ S_x & I_{xy} & I_x \end{bmatrix}^{-1} \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$



Simplifications

- If x and y are centroidal axes, and N is absent, then

$$\begin{Bmatrix} \frac{d^2u}{dz^2} \\ \frac{d^2v}{dz^2} \end{Bmatrix} = \frac{-1}{E}\begin{bmatrix} I_y & I_{xy} \\ I_{xy} & I_x \end{bmatrix}^{-1} \begin{Bmatrix} M_y \\ M_x \end{Bmatrix}$$

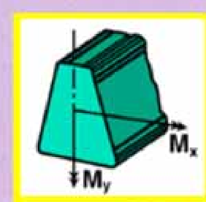


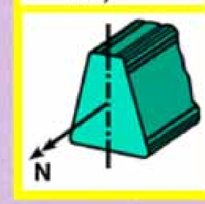
Calculations of Displacements

Simplifications

- If x and y are centroidal axes, and N is absent, then

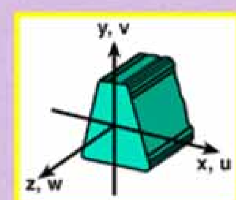
$$\begin{Bmatrix} \dfrac{d^2u}{dz^2} \\ \dfrac{d^2v}{dz^2} \end{Bmatrix} = \frac{-1}{E} \begin{bmatrix} I_y & I_{xy} \\ I_{xy} & I_x \end{bmatrix}^{-1} \begin{Bmatrix} M_y \\ M_x \end{Bmatrix}$$

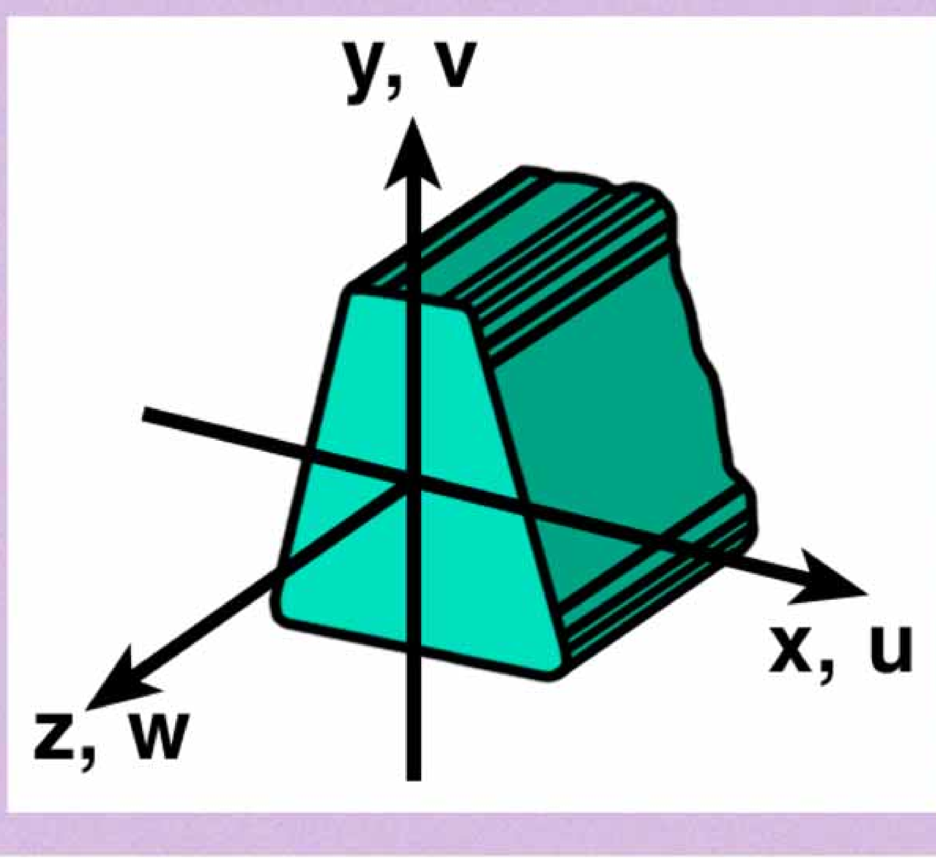




- If x and y are centroidal principal axes, and N is absent, then

$$\begin{Bmatrix} \dfrac{d^2u}{dz^2} \\ \dfrac{d^2v}{dz^2} \end{Bmatrix} = \frac{-1}{E} \begin{Bmatrix} \dfrac{M_y}{I_y} \\ \dfrac{M_x}{I_x} \end{Bmatrix}$$



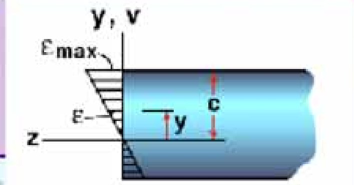


Calculations of Displacements

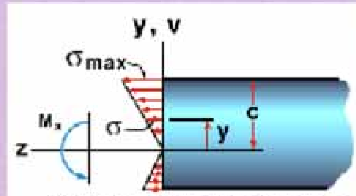
Case of Transverse Bending

- Governing equation for the elementary theory of beams:

$$\frac{d^2}{dz^2}\left(EI_x \frac{d^2 v}{dz^2}\right) = p_y$$



Normal strain variation (profile view)



Bending stress variation (profile view)

- For uniform beams

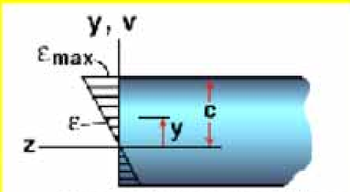
Calculations of Displacements

Case of Transverse Bending

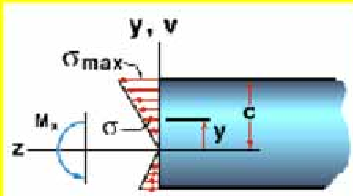
- For uniform beams

$$EI \frac{d^4 v}{dz^4} = p$$

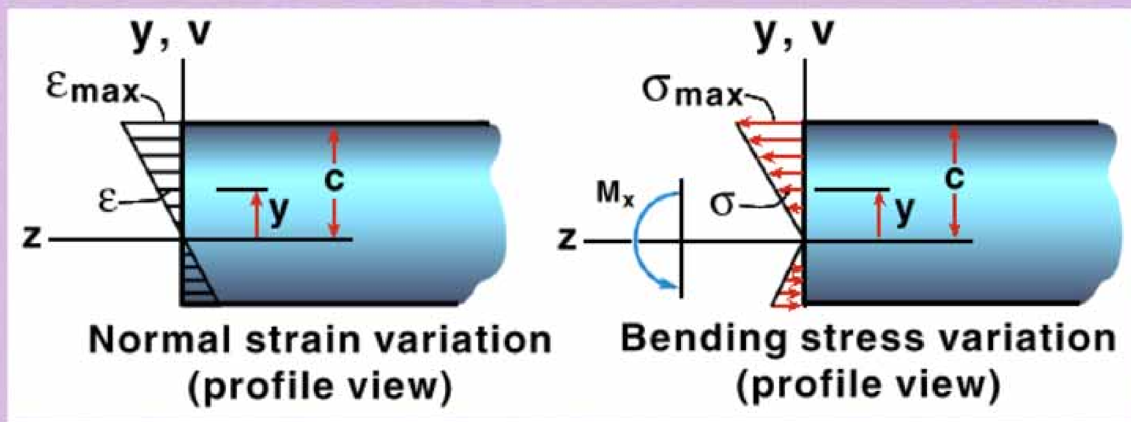
where subscripts x and y have been dropped for convenience.



Normal strain variation (profile view)



Bending stress variation (profile view)



Calculations of Displacements

Case of Transverse Bending

- *Successive integration of the differential equation*

Transverse shear

$$EI\frac{d^3v}{dz^3} = \int_0^z p\,dz + c_1 = -V$$

Calculations of Displacements

Case of Transverse Bending

Bending Moment

$$EI\frac{d^2v}{dz^2} = \int_0^z\int_0^z p\,dz\,dz + c_1z + c_2$$
$$= -M$$

Calculations of Displacements

Case of Transverse Bending

Slope

$$EI\frac{dv}{dz} = \int_0^z\int_0^z\int_0^z p\,dz\,dz\,dz$$
$$+ \frac{1}{2}c_1z^2 + c_2z + c_3$$

Calculations of Displacements

Case of Transverse Bending

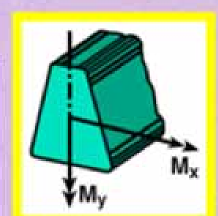
Transverse Displacement

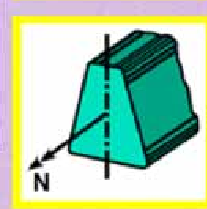
$$EI\ v = \int_0^z \int_0^z \int_0^z \int_0^z p\, dz\, dz\, dz\, dz + \frac{1}{6}c_1 z^3 + \frac{1}{2}c_2 z^2 + c_3 z + c_4$$

Calculations of Displacements

- Case of combined $\boxed{N, M_x, M_y}$

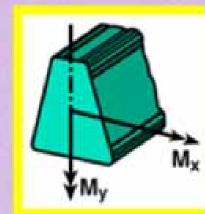
$$\sigma = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} A & S_y & S_x \\ S_y & I_x & I_{xy} \\ S_x & I_{xy} & I_x \end{bmatrix}^{-1} \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$

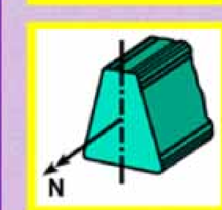




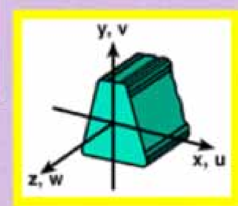
Calculations of Displacements

- Displacement equations



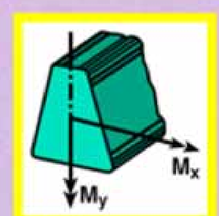


$$\begin{Bmatrix} \varepsilon_o \\ -\dfrac{d^2u}{dz^2} \\ -\dfrac{d^2v}{dz^2} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} A & S_y & S_x \\ S_y & I_y & I_{xy} \\ S_x & I_{xy} & I_x \end{bmatrix}^{-1} \begin{Bmatrix} N \\ M_y \\ M_x \end{Bmatrix}$$

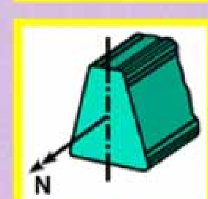


Calculations of Displacements

- If x,y are centroidal principal axes



$$\boxed{\frac{dw}{dz} = \frac{N}{EA}}$$



$$\boxed{-\frac{d^2u}{dz^2} = \frac{M_y}{EI_y}} \qquad \boxed{-\frac{d^2v}{dz^2} = \frac{M_x}{EI_x}}$$

w is the displacement of the axis of the beam in the z direction